Effects of diversity and procrastination in priority queuing theory: The different power law regimes

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Empirical analyses show that after the update of a browser, or the publication of the vulnerability of a software, or the discovery of a cyber worm, the fraction of computers still using the older browser or software version, or not yet patched, or exhibiting worm activity decays as a power law $\sim 1/t^{\alpha}$ with $0<\alpha\leq 1$ over a time scale of years. We present a simple model for this persistence phenomenon, framed within the standard priority queuing theory, of a target task which has the lowest priority compared to all other tasks that flow on the computer of an individual. We identify a "time deficit" control parameter β and a bifurcation to a regime where there is a nonzero probability for the target task to never be completed. The distribution of waiting time $\mathcal{T}$ until the completion of the target task has the power law tail $\sim 1/t^{1/2}$, resulting from a first-passage solution of an equivalent Wiener process. Taking into account a diversity of time deficit parameters in a population of individuals, the power law tail is changed into $1/t^{\alpha}$, with $\alpha \in (0.5,\infty)$, including the well-known case $1/t$. We also study the effect of "procrastination," defined as the situation in which the target task may be postponed or delayed even after the individual has solved all other pending tasks. This regime provides an explanation for even slower apparent decay and longer persistence.

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I. INTRODUCTION

There is growing evidence that remarkably robust quantitative and sometimes universal laws describe the behavior of humans in society. Consider a typical individual, who is subjected to a flow of information and requested tasks, in the presence of time, energy, regulatory, social, and monetary constraints. Such an individual will respond by a sequence of decisions and actions, which themselves contribute to the flow of influences spreading to other individuals. A recently developed approach to unravel at least a part of this complex ballet consists in quantifying the waiting time distribution between triggering factor and response performed by humans, which has been found in many situations to be a power law $\mathrm{pdf}(t)\sim 1/t^{p}$ with an exponent p less than 2. As a consequence, the mathematical expectation of the waiting time between consecutive events is infinite, which embodies the notion of a very long persistence of past influences. This power law behavior has been documented quantitatively for the distribution of waiting times until an email message is answered [1], for the time intervals between consecutive emails sent by a single user and time delays for email replies [2], for the waiting time between receipt and response in the correspondence of Darwin and of Einstein [3], and for the waiting times associated with other human activity patterns which extend to web browsing, library visits, and stock trading [4].

A related measure concerns the rate of activity following a shock, a perturbation, an announcement, and so on that impacts a given social system. For instance, measures of media coverage after a large geopolitical event decay approximately as a power law of time since the occurrence of the event [5]. The rate of downloads of papers from a website after a media coverage also follows a power law decay [6,7]. The rate of book sales following an advertisement or large media exposition decays as a power law of the time since that event [8,9]. The rate of video views on YouTube decays also as a power law after peaks associated with media exposure [10]. Reference [11] argues that many other systems are described by a similar behavior.

These two measures of human reactions, (i) distribution of waiting times between triggering factor and response and (ii) rate of activity in response to a "shock," are related. This fact has been exemplified by the rate of donations following the tsunami that occurred on December 26, 2004 [12]. A donation associated with this event can be considered as a task that was triggered (but not necessarily executed) on that day simultaneously for a large population of potential donors. This task competes with many others associated with the jobs, private lives, and other activities of each individual in the entire population. The social experiment provided by the tsunami illustrates a general class of experiments in which the same "singular task" is presented at approximately the same time to all potential actors (here the donors), but the priority value of this singular task can be expected to be widely distributed among different individuals. Since the singular task has been initiated at nearly the same time for all individuals, the activity (number of donations) at a time t after this initiation time is then simply equal to $N\times \mathrm{pdf}(t)$, where N is the number of individuals who will eventually act (donate) in the population and $\mathrm{pdf}(t)$ is the previously defined distribution of waiting times until a task is executed.

These observations have been rationalized by priority queuing models that describe how the flow of tasks falling on (and/or self-imposed by) humans are executed using priority ranking [2–4]. Assuming that the average rate λ of task

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arrivals is larger than the average rate μ for executing them and using a standard stochastic queuing model wherein tasks are selected for execution on the basis of random continuous priority values, Grinstein and Linsker [13] derived the exact overall probability per unit time, pdf(t), that a given task sits in the queue for a time t before being executed

$$\mathrm{pdf}(t) \sim \frac{1}{t^{3/2}}, \quad \text{for} \quad \mu \le \lambda. \tag{1}$$

Grinstein and Linsker [14] showed that the distribution (1) is independent of the specific shape of the distribution of priority values among individuals. The value of the exponent $p=3/2$ is compatible with previously reported numerical simulations [2–4] and with most but not all of the empirical data.

Our present theoretical study can be considered both as a pedagogical simplification and an extension of Grinstein and Linsker [13,14], with the goal of exploring different mechanisms explaining the deviations of the exponent p from its canonical value 3/2. In particular, we reveal the fundamental statistical origin of the power law (1) with exponent 3/2 as nothing but a first-passage problem of an underlying random walk [15]. The initial motivation for the present study came from recent quantitative empirical studies [16,17,19] on the time decay of the use of outdated browsers on the internet and of the remaining detectable and surprisingly significant activity on the internet of the Blaster worm since 2003 up to present [18,19]. The activity in these systems reveals the equivalent of the survival distribution of browsers or of computers which have not yet been updated or patched (in the language of priority tasks, this is the fraction of all entities which have not yet accomplished the task). These activities are found to decay as $\sim 1/t^{\alpha}$, with $\alpha \equiv p-1$, where the exponent is one unit less than for pdf(t) since it describes the decay of the complementary cumulative (also known as the survival) distribution of entities that have not yet acted. The new element is that α is found to be different from 1/2, sometimes smaller, while it is larger in other cases. Here, we ask what could be the simplest explanations for such behaviors.

The structure of the paper is as follows. Section II presents the model of a target task which has the lowest priority compared to all other tasks that flow on the "shoulders" (or computer) of an individual. The distribution of waiting time $\mathcal{T}$ until the completion of the target task is formulated as a first-passage time of an approximately equivalent Wiener process with drift. There is a control parameter, which we call the "time deficit" parameter β, that is proportional to the drift of the associated Wiener process. It is proportional to the difference between the average time $\langle \eta \rangle$ to complete a nontarget task and the average time interval $\langle \tau \rangle$ between nontarget task arrivals: $\beta \propto \langle \eta \rangle - \langle \tau \rangle$. For small β's, the probability density function (pdf) $q(t)$ of $\mathcal{T}$ has a power law tail $1/t^{1+\alpha}$, with exponent $\alpha=1/2$. Its corresponding complementary cumulative distribution $Q(t)$ exhibits a bifurcation as a function of β. For $\beta<0$ but close to 0, $Q(t)\sim 1/t^{\alpha}$ and tends to zero at long time. For $\beta>0$, $Q(t)\sim Q_{\infty}+C/t^{\alpha}$, where Q_{∞} is the nonzero probability that the target task is never completed. Section III extends the preceding analysis to a population of individuals with different time deficit parameters β. We distinguish between regular distributions around $\beta=0$ and nonregular ones. For the former, the exponent α is changed into the value 1 by the effect of heterogeneity. For distributions of β that allow for positive values, the survival distribution $Q(t)$ exhibits again a nonzero asymptotic limit at large times. For nonregular distributions of β, the exponent α is found to be continuously tunable from 1/2 to $+\infty$. Section IV introduces the mechanism of "procrastination," defined as the situation in which the target task may be postponed or delayed even after the individual has solved all other pending tasks. In the limit where the procrastination inclination is large and the time deficit parameter is close to zero, we find that the pdf $q(t)$ of $\mathcal{T}$ exhibits a new much slower power law tail $\sim 1/t^{1-\alpha}$, with $\alpha=1/2$ in the regular case. The survival distribution $Q(t)$ is characterized by a slow crossover to the asymptotic power law $\sim 1/t^{\alpha}$. Section V concludes.

II. MODEL AND STANDARD SOLUTIONS

A. Formulation in terms of a specific target task with the lowest priority

Let us assume for definiteness, but without loss of generality, that the target task is identified at the origin of time $t=0$. For concreteness, we will frame our discussion by using the examples of the task of updating your browser version on your computer to the newly available version. Another example is the task of patching a software after its vulnerability has been disclosed and its patch has been made freely available. Our goal is to derive the distribution of waiting times or, equivalently, the dependence with time of the fraction of the population that has not yet performed the task.

Starting with the classical theory of a prioritized queue, we assume that the target task has the lowest priority among all other user's tasks. In other words, the users consider updating their browser or patching their softwares as doable only after all their other tasks have been addressed. This captures the casual empirical observation that computer users are often reluctant to interrupt their work, social chatting and blogging, games, and other activities on their computer for an update or patch that often requires a complete shutdown and restart.

The time at which the target task is performed is denoted $\mathcal{T}$: it corresponds to the time interval over which the user has been busy doing other things. We therefore refer to it as the "busy time duration." By definition of $\mathcal{T}$, for any $t\in(0,\mathcal{T})$, there are still other unsolved tasks that requires the attention of the individual, while at the instant $t=\mathcal{T}$, all tasks that arose earlier have been solved. In the present section, we assume that, once freed of other preoccupations at time $\mathcal{T}$, the individual who has been presented with the target task at $t=0$ will finally perform it immediately. In Sec. IV, we investigate another situation in which, once free of other constraints, the user nevertheless procrastinates. Then, new tasks may appear in the meantime, leading to further delays in the completion of the target task. This procrastination mechanism leads to new slower decay laws and interesting crossover regimes. But, with the present assumption that the target task is addressed as soon as the user is free of other tasks, it holds true that the complementary cumulative distribution function $Q(t)$ of waiting times until the update of the browser coincides with the probability that the busy time duration is larger than the given instant t,

$$Q(t) = \Pr\{\mathcal{T} > t\}. \tag{2}$$

We now analyze in detail the components contributing to the busy time duration $\mathcal{T}$. Consider first all the tasks which were present before $t=0$ when the new target task was first presented to the individual and which have not yet been completed. Let us assume that a time η_0 is still needed after $t=0$ to solve these tasks. Then, in the scenario in which no new tasks occur, we have

$$Q(t) = \Pr\{\eta_0 > t\}. \tag{3}$$

We consider now all the other scenarios in which new tasks may fall on the shoulders of the individual after $t=0$. Specifically, let us assume that the number of such new tasks grows with t according to some staircase function $n(t)$, which increases by one unit at discrete occurrence times

$$0 < T_1 < T_2 \cdots < T_{n(t)} < t. \tag{4}$$

We denote the time needed by the individual to solve the kth task as η_k. Then, by definition of the busy time duration $\mathcal{T}$, $Q(t)$ is given by the probability of the chain of events $\cap_{k=0}^{n(t)} \mathcal{A}_k$,

$$Q(t) = \Pr\{\mathcal{T} > t\} \equiv \Pr[\bigcap_{k=0}^{n(t)} \mathcal{A}_k], \tag{5}$$

where

$$\mathcal{A}_k = \eta_0 + W(k) - T_{k+1} > 0 \tag{6}$$

and

$$W(k) = \sum_{i=1}^{k} \eta_i, \quad W(0) = 0. \tag{7}$$

In Eq. (6), it is understood that $T_{n(t)+1} \equiv t$

Expression (5) shows that the chain $\cap_{k=0}^{n(t)} \mathcal{A}_k$ of events $\mathcal{A}_k$ defined by Eq. (6) determines $Q(t)$ completely. It is thus important to have a detailed understanding of it. First, the event with index 0 is nothing but

$$\mathcal{A}_0 \equiv \eta_0 > T_1, \tag{8}$$

which corresponds to the scenario in which the time η_0 that the individual needs to solve all tasks stored up to $t=0$ is larger than the time T_1 at which the first new task appears after the time $t=0$ when the target task was assigned. Since the individual is still solving other tasks, she cannot perform the target tasks before the arrival of the first new task at T_1. The event

$$\mathcal{A}_1 \equiv \eta_0 + \eta_1 > T_2 \tag{9}$$

represents the scenario in which the individual is still solving the tasks that were not yet finished before $t=0$ or the new task that appeared at time T_1 when the second task occurs at time T_2. Again, the individual is busy until time T_2 and cannot address the target task. The set of events $\mathcal{A}_k$ for all k's up to $n(t)$ follows the same structure, so that the individual has still not had time to address the target task at time $t < \mathcal{T}$. Figure 1 illustrates this chain $\cap_{k=0}^{n(t)} \mathcal{A}_k$ of events.

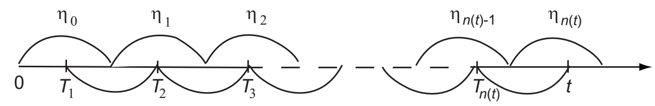

FIG. 1. Schematic illustration of the chain of events $\cap_{k=0}^{n(t)} \mathcal{A}_k$, leading to the occurrence of $\mathcal{T} > t$, where $\mathcal{T}$ is the "busy time duration" until the target task is performed.

We now introduce the auxiliary stochastic process

$$V(k) = \sum_{i=1}^{k} (\eta_k - \tau_k), \quad V(0) = 0, \tag{10}$$

where

$$\tau_k = T_{k+1} - T_k \tag{11}$$

is the time interval separating the occurrence of the kth and $(k+1)$th tasks arising after $t=0$. It is also convenient to define

$$\eta_0' = \eta_0 - T_1 \tag{12}$$

as the time needed to complete all tasks stored up to $t=0$ when the first new task occurs at time $T_1 > 0$. The case $\eta_0' \leq 0$ is excluded as it would correspond to the completion of the target task at time η_0 before the arrival of the first task at time T_1. In this case, all subsequent tasks become irrelevant. We also assume for simplicity that η_0' is a fixed deterministic value (we will relax this condition later on), so that

$$\Pr[\bigcap_{k=0}^{n(t)} \mathcal{A}_k] = \Pr[V(k) > -\eta_0'], \quad \text{for any} \quad k \in [0, n(t)]. \tag{13}$$

The sought complementary distribution $Q(t)$ defined by expression (5) is therefore given by

$$Q(t) = \Pr\{V(k) > -\eta_0' : k \in [0, n(t)]\}. \tag{14}$$

Figure 2 shows a typical realization of the stochastic process $V(k)$ defined by expression (10) over a time interval in which the inequality $V(k) > -\eta_0'$ defining $Q(t)$ in Eq. (14) holds.

B. Approximation in terms of a Wiener process (random walk) with drift

We assume that the two sequences $\{\eta_k\}$ and $\{\tau_k\}$ are made of independent identically distributed (i.i.d.) random numbers, with mean and variance, respectively, equal to $\langle \eta \rangle$, σ_η^2, and $\langle \tau \rangle$, σ_τ^2. In order to obtain the asymptotical properties of $Q(t)$ given by Eq. (14) at large times $t \gg \langle \tau \rangle$, we apply the law of large numbers (LLN) that justifies replacing the random number $n(t)$ in Eq. (14) by its *mean*,

$$n(t) \approx \langle n(t) \rangle = \theta \equiv t / \langle \tau \rangle, \tag{15}$$

leading to the following asymptotically exact expression for $Q(t)$:

$$Q(t) \approx \Pr[V(k) > -\eta_0' : k \in (0, \theta)]. \tag{16}$$

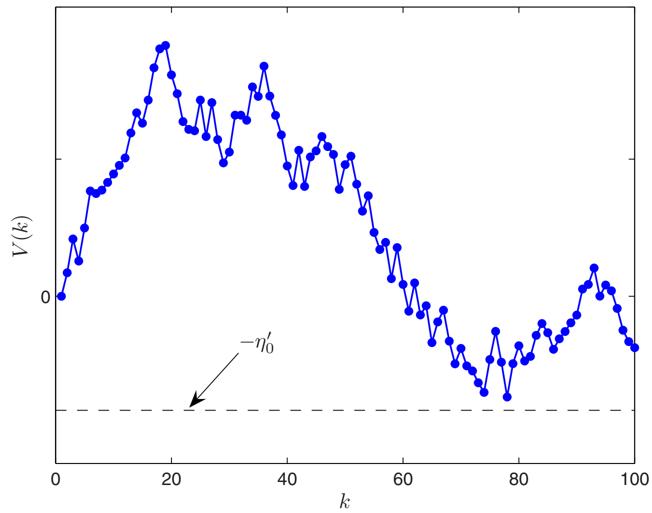

FIG. 2. (Color online) Typical realization of the stochastic process $V(k)$ defined by expression (10) as a function of the discrete argument k indexing the successive tasks appearing after $t=0$ at which the target task has been initiated. The realization shown here obeys the inequality $V(k) > -\eta_0'$ defining $Q(t)$ in Eq. (14) over the whole time interval shown.

For large times and thus large θ's, the central limit theorem (CLT) ensures that the process $V(k)$ defined by Eq. (10) can be interpreted as the discrete version of a Wiener process with drift, with the following mean value and variance:

$$\langle V(k)\rangle = k\cdot(\langle\eta\rangle - \langle\tau\rangle), \quad \mathrm{Var}[V(k)] = \sigma^2\cdot k, \quad \sigma^2 = \sigma_\eta^2 + \sigma_\tau^2. \tag{17}$$

Moreover, for $\theta \gg 1$, we can replace $V(k)$ by its continuous limit, in the form of the standard Wiener process with drift, depending on the *continuous* argument k. Accordingly, the pdf $f(v;k)$ of the continuous stochastic process $V(k)$ satisfies the diffusion equation

$$\frac{\partial f(v;k)}{\partial k} + (\langle\eta\rangle - \langle\tau\rangle)\frac{\partial f(v;k)}{\partial v} = \frac{\sigma^2}{2}\frac{\partial^2 f(v;k)}{\partial v^2}, \tag{18}$$

supplemented by the initial condition

$$f(k;0) = \delta(v). \tag{19}$$

Recall that $f(v;k)dv$ is the probability of finding $V(k)$ between v and $v+dv$ at "time" k.

From the theory of Wiener processes, and within the asymptotically exact continuous limit just described, it follows that the probability given Eq. (16) is given by

$$Q(t) = \int_{-\eta_0'}^{\infty} f(v;\theta|\eta_0')dv, \tag{20}$$

where $f(v;k|\eta_0')$ is the solution of the diffusion equation (18) satisfying the initial condition (19) and the additional absorbing boundary condition

$$f(v;k|\eta_0')|_{v=-\eta_0'} = 0. \tag{21}$$

The solution of the initial-boundary problem (18), (19), and (21) is

$$f(v;k|\eta_0') = g[v - (\langle\eta\rangle - \langle\tau\rangle)k;k] - \exp\left[-\frac{2(\langle\eta\rangle - \langle\tau\rangle)\eta_0'}{\sigma^2}\right] \times g[v - (\langle\eta\rangle - \langle\tau\rangle)k + 2\eta_0';k], \tag{22}$$

where

$$g(v;k) = \frac{1}{\sqrt{2\pi k}\sigma}\exp\left(-\frac{v^2}{2\sigma^2 k}\right). \tag{23}$$

Substituting expression (22) into Eq. (20) yields

$$Q(t) \approx \frac{1}{2}\left[1 + \mathrm{erf}\left(\frac{\gamma+\delta\theta}{\sqrt{2\theta}}\right) - e^{-2\delta\gamma}\,\mathrm{erfc}\left(\frac{\gamma-\delta\theta}{\sqrt{2\theta}}\right)\right], \tag{24}$$

with the following notations:

$$\gamma = \frac{\eta_0'}{\sigma_\eta}, \quad \delta = \frac{\langle\eta\rangle - \langle\tau\rangle}{\sigma}, \quad \theta = \frac{t}{\langle\tau\rangle}. \tag{25}$$

The corresponding pdf is

$$q(t) \equiv -\frac{dQ(t)}{dt} = \frac{1}{\langle\tau\rangle}\frac{\gamma}{\sqrt{2\pi}\theta^{3/2}}\exp\left(-\frac{(\delta\theta+\gamma)^2}{2\theta}\right). \tag{26}$$

In the analysis that follows, the key role played by the parameter δ defined in (25) warrants further interpretation. Let us assume that the occurrence of new tasks at the times $\{T_k\}$ defined by Eq. (4) is a Poisson flow with rate λ. Similarly, we assume that the completion of tasks at the times $\{W(k)\}$ defined by Eq. (7) is also a Poissonian flow with rate μ. We thus have

$$\langle\tau\rangle = \sigma_\tau = \frac{1}{\lambda}, \quad \langle\eta\rangle = \sigma_\eta = \frac{1}{\mu}, \tag{27}$$

where $\langle\tau\rangle$ is the mean time between arriving tasks and $\langle\eta\rangle$ is the mean completion time of the tasks. Accordingly, the parameter δ is equal to

$$\delta = \frac{\varepsilon - 1}{\sqrt{\varepsilon^2+1}}, \quad \text{where} \quad \varepsilon = \lambda\langle\eta\rangle = \frac{\lambda}{\mu}. \tag{28}$$

When

$$\langle\tau\rangle = \langle\eta\rangle \Rightarrow \delta = 0 \Rightarrow \varepsilon = 1, \tag{29}$$

the rate of new task arrivals is equal to the rate of solving them. This *balanced* condition corresponds to a zero drift in the associated Wiener process and plays a crucial role in the generation of power laws in the distribution of waiting times. When positive, the parameter δ quantifies the "time deficit" that is missing on average per task in order for the individual to finally be able to complete the target task. For a negative time deficit ($\delta < 0$), the target task is almost surely performed in finite time as we show below.

It is instructive to analyze separately the complementary cumulative distribution $Q(t)$ given by Eq. (24) and its corresponding pdf $q(t)$ of the waiting time for the target task to be done (browser upgrade or software patched) given by Eq. (26). We will discuss different situations in which $Q(t)$ and $q(t)$ are described by asymptotic power laws

$$Q(t) \sim Q_\infty + \theta^{-\alpha} \Leftrightarrow q(t) \sim \theta^{-\alpha-1}, \tag{30}$$

where Q_∞ may be nonzero in some interesting cases to be discussed below.

C. Derivation of the power law pdf $q(t)$ of waiting times till the completion of the target task

We first rewrite the pdf $q(t)$ given by Eq. (26) in a form more convenient for its analysis. For this, we note that $q(t)$ is controlled by two characteristic scales

$$\theta_\gamma = \frac{\gamma^2}{2}, \quad \theta_\delta = \frac{2}{\delta^2}. \tag{31}$$

For definiteness, consistent with our previous assumption that η_0' is constant, we take γ (i.e., θ_γ) to be constant. We then explore the behavior of $q(t)$ for different values of δ (i.e., θ_δ). It is convenient to introduce the new variable

$$\rho = \frac{\theta}{\theta_\gamma} = \frac{2\theta}{\gamma^2} \tag{32}$$

and the parameter

$$\beta = \sqrt{\frac{\theta_\gamma}{\theta_\delta}} = \frac{1}{2}\gamma\delta. \tag{33}$$

The dimensionless pdf

$$\kappa(\rho;\beta) = \frac{\gamma^2}{2}\sqrt{\pi}\langle\tau\rangle q(t) \tag{34}$$

takes the form

$$\kappa(\rho;\beta) = \frac{1}{\rho^{3/2}}\exp\left(-\frac{(\beta\rho+1)^2}{\rho}\right). \tag{35}$$

The dependence of $\kappa(\rho;\beta)$ as a function of ρ is qualitatively different for $\beta \ll 1$ ($\theta_\gamma \ll \theta_\delta$) and for $\beta \gtrsim 1$ ($\theta_\gamma \gtrsim \theta_\delta$). For $\beta \gtrsim 1$, $\kappa(\rho;\beta)$ does not exhibit any power law asymptotic regime, not even in an intermediate domain of ρ. In contrast, for

$$\beta \ll 1 \Leftrightarrow \frac{1}{2}\gamma\delta \ll 1, \tag{36}$$

the pdf $\kappa(\rho;\beta)$ possesses the intermediate power asymptotic regime

$$\kappa(\rho;\beta) \approx \rho^{-3/2}, \quad 1 \lesssim \rho \lesssim \beta^{-2}, \tag{37}$$

which is replaced, for larger ρ, by the exponential decay

$$\kappa(\rho;\beta) \approx \rho^{-3/2}e^{-\beta^2\rho}. \tag{38}$$

Note that the function $\kappa(\rho;\beta)$ is simply the pdf of the first return to the absorbing boundary condition defined in Eq. (21) (the target task is performed) of the Wiener process with drift with the characteristics (17) [15]. In the balanced case (29)

$$\delta = 0 \Rightarrow \beta^{-2} = \infty,$$

the power law (37) holds for any $\rho \gtrsim 1$.

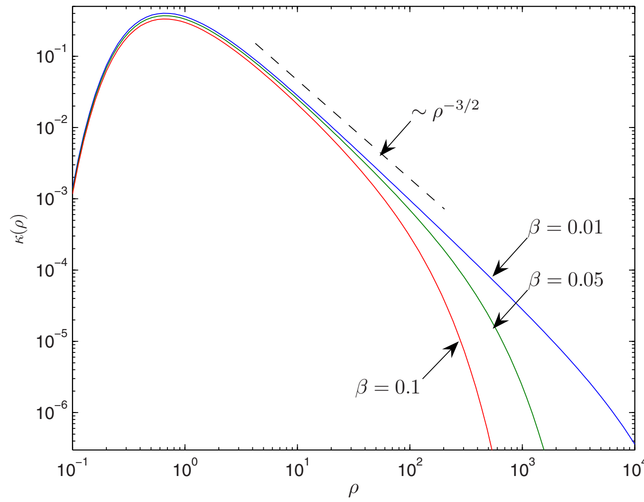

FIG. 3. (Color online) Dependence of $\kappa(\rho;\beta)$ given by Eq. (35) as a function of ρ for three values of the normalized drift parameter β=0.01;0.05;0.1. Dashed straight line corresponds to the pure power law $\sim\rho^{-3/2}$. The larger the value of β, the narrower the interval in ρ for which the intermediate power asymptotic regime holds.

Figure 3 illustrates the crossover of $\kappa(\rho;\beta)$ from the intermediate power law asymptotic regime (37) and the exponential tail (38) for three increasing values of the normalized drift parameter β.

D. Derivation of the survival distribution $Q(t)$ of waiting times until the completion of the target task

The complementary cumulative distribution $Q(t)$ given by Eq. (24) can be rewritten in the following form, which is more convenient for the analysis of its asymptotic behavior:

$$Q(t) \equiv \mathcal{Q}(\rho;\beta) = \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{1+\beta\rho}{\sqrt{\rho}}\right) - e^{-4\beta}\operatorname{erfc}\left(\frac{1-\beta\rho}{\sqrt{\rho}}\right)\right]. \tag{39}$$

This allows us to show that the function $\mathcal{Q}(\rho;\beta)$ has a qualitatively different behavior at $\rho \to \infty$ for $\beta > 0$ and for $\beta < 0$. This can be seen from the corresponding limits of $\mathcal{Q}(\rho;\beta)$:

$$\lim_{\rho\to\infty} \mathcal{Q}(\rho;\beta) = \begin{cases} Q_\infty(\beta), & \beta > 0 \\ 0, & \beta < 0, \end{cases} \quad Q_\infty(\beta) = 1 - e^{-4\beta}. \tag{40}$$

For $\beta > 0$, $Q(t)$ tends to a strictly *positive limit* $Q_\infty(\beta) > 0$ as $\rho \to +\infty$. For $\beta < 0$, $\mathcal{Q}(\rho;\beta)$ tends to zero as $\rho \to +\infty$. These two limits have the following simple probabilistic interpretations. For $\beta > 0$, $\langle\eta\rangle > \langle\tau\rangle$: the average time needed to complete a task is larger than the average interarrival times between new incoming tasks. As a consequence, there is strictly positive probability $Q_\infty(\beta) > 0$ that the target task will never be completed. In the language of the drifting Wiener process with characteristics (17), in the presence of a positive drift, there is a finite probability $Q_\infty(\beta)$ for the Wiener process to escape to infinity (the target task is never completed)

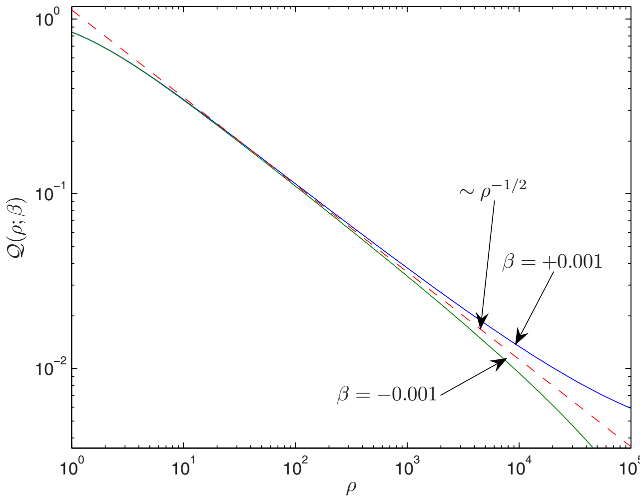

FIG. 4. (Color online) Complementary cumulative distribution $\mathcal{Q}(\rho;\beta)$ for $\beta=\pm 10^{-3}$ as a function of the normalized waiting time ρ for completing the target task, demonstrating the qualitatively different asymptotic behavior of $\mathcal{Q}(\rho;\beta)$ for $\beta>0$ and for $\beta<0$. Dashed straight line shows the asymptotic power law (41) corresponding to the balanced case $\beta=0$.

and a probability $1-Q_\infty(\beta)$ for being captured at the absorbing boundary defined in Eq. (21) (the target task is performed). In contrast, for $\beta<0$, $\langle\eta\rangle<\langle\tau\rangle$: the individual will almost surely complete the target task (update her browser or patch her software) in finite time. Thus, for $\beta<0$, $Q_\infty(\beta)=0$. Crossing the value $\beta=0$ is analogous to a phase transition or bifurcation characterized by the order parameter $Q_\infty(\beta)$ varying as a function of the control parameter β: for $\beta<0$, the order parameter is zero and it bifurcates to a nonzero value for $\beta>0$.

Equation (39) expressed for the balanced case $\beta=0$ yields the following power law asymptotic regime:

$$\mathcal{Q}(\rho;0) \approx \frac{2}{\sqrt{\pi\rho}}, \quad \rho \gtrsim 1, \tag{41}$$

corresponding to the power law exponent $\alpha=1/2$ as defined in Eq. (30). For $|\beta|\gtrsim 1$, $\mathcal{Q}(\rho;\beta)$ does not exhibit any power law asymptotic regime, not even in an intermediate domain of ρ. For $|\beta|\ll 1$, the power law (41) holds as an intermediate asymptotic regime in the interval

$$1 \lesssim \rho \lesssim \beta^{-2}. \tag{42}$$

For $\rho\gg\beta^{-2}$, the intermediate power law asymptotic regime (41) crosses over to an exponential decay converging to 0 for $\beta<0$ or to $Q_\infty(\beta)>0$ given in Eq. (40) for $\beta>0$,

$$\mathcal{Q}(\rho;\beta) \approx \frac{1}{\beta^2\sqrt{\pi}\rho^{1/2}}\exp(-2\beta-\beta^2\rho) + \begin{cases} Q_\infty(\beta)>0, & \beta>0 \\ 0, & \beta<0. \end{cases} \tag{43}$$

Figure 4 plots the dependence of $\mathcal{Q}(\rho;\beta)$ as a function of ρ for $\beta=\pm 0.001$, which illustrates the qualitatively different behavior of $\mathcal{Q}(\rho;\beta)$ for $\beta>0$ and for $\beta<0$.

III. DISTRIBUTIONS OF THE TIME DEFICIT PARAMETER LEADING TO DIFFERENT POWER LAW EXPONENTS AND REGIMES

A. Regular distribution of the normalized time deficit parameter β around the origin

1. Qualitative justification of the form of the normalized time deficit parameter β

As shown in the previous section, proximity to the balance condition (29) is essential for the power laws (37) or (41) to hold over an intermediate asymptotic region sufficiently large to be observable (at least over 1 to 2 decades in time). In the present theory, the time deficit parameter δ, or equivalently its normalized version β, is exogenously given. In reality, δ (β) embodies the interplay between the subtle processes of task formation, prioritization, and the efforts undertaken to solve the tasks that each individual continuously adjusts. We conjecture that users of browsers and of softwares adapt approximately but not exactly, of course, to the balance condition (29). This is done, for instance, by being selective among the flow of tasks (by deleting needless incoming emails or ignoring some superfluous tasks) and/or by adapting the time allocated to solving tasks, so that the mean time $\langle\eta\rangle$ needed to solve a given problem is approximately equal to the mean time interval $\langle\tau\rangle$ between subsequent arriving tasks. Suppose for instance that $\langle\eta\rangle>\langle\tau\rangle$. In this case, the individual is not able to face the flow of incoming tasks and a boundless number of tasks pile up, suggesting a nonsustainable regime either for the computer or its user. In the opposite case $\langle\eta\rangle<\langle\tau\rangle$, the individual sits idle for a significant fraction of her time. By enlarging the definition of what is meant by "task" to include other activities, including the recreational activities that arguably constitute a significant part of the utility or pleasure driving individuals, it is clear that the case $\langle\eta\rangle<\langle\tau\rangle$ is not realistic as a sustained regime. We also conjecture that the adjustment process leading to the convergence of $\langle\tau\rangle$ towards $\langle\eta\rangle$ and vice versa may describe the general problem of task flow versus their solutions, beyond the specific problem of browser update and software patching discussed here, to encompass the general balance of human activities. We thus believe that the results presented here are of broader interest and may help understand the general statistical properties of the time allocation of humans.

Following these arguments, while individuals can be expected to adjust toward the balance condition (29), it is unlikely that all humans will do so accurately. Indeed, in many systems subjected to noise in which state-dependent control actions are performed, the control parameter never settles but continues to fluctuate around the target parameter [20–22]. Therefore, we propose that the population of browser and software users can be described by a distribution of time deficit parameters δ (β) which is centered on 0. We thus take into account the mentioned fluctuations by considering δ as random variable with some pdf $\phi(\delta)$. The idealized balance condition $\delta=0$ is replaced by the more realistic *mean balance condition*

$$\langle \delta \rangle = \int_{-\infty}^{\infty} \delta \phi(\delta) d\delta = 0. \quad (44)$$

Assuming that γ is some deterministic constant as above, then the mean balance condition (44) is equivalent to

$$\langle \beta \rangle = 0. \quad (45)$$

2. Derivation of the pdf $q(t)$ for a Gaussian distribution of the normalized time deficit parameter β

In order to obtain concrete quantitative predictions, let us first consider that the pdf $\psi(\beta)$ of the random variable β is a Gaussian law centered on 0 and with standard deviation β_0,

$$\psi(\beta;\beta_0) = \frac{1}{\sqrt{2\pi}\beta_0}\exp\left(-\frac{\beta^2}{2\beta_0^2}\right). \quad (46)$$

The idealized balance condition $\beta=0$ is recovered in the limit $\beta_0 \to 0$ for which the Gaussian pdf (46) tends to the Dirac function $\psi(\beta)=\delta(\beta)$. With the choice (46), the mean balance condition (45) holds by construction.

Amazingly, it turns out that the fluctuations of the parameter β, which satisfy the mean balance condition $\langle\beta\rangle=0$, drastically change the asymptotic form of the distribution $Q(t)$ and its associated pdf $q(t)$, compared to the idealized case in which the balance condition $\beta=0$ holds exactly for each individual at all times. Let us first analyze this effect on the pdf of waiting times for the completion of the target task, which is now given, in its normalized version by the weighted average with respect to β of $\kappa(\rho;\beta)$ given by Eq. (35)

$$\bar{\kappa}(\rho) = \int_{-\infty}^{\infty} \kappa(\rho;\beta)\psi(\beta)d\beta. \quad (47)$$

Using the Gaussian distribution (46), this yields

$$\bar{\kappa}(\rho;\beta_0) = \frac{1}{\rho^{3/2}\sqrt{1+2\beta_0^2\rho}}\exp\left(-\frac{2\beta_0^2}{\rho(1+2\beta_0^2\rho)}\right), \quad (48)$$

which has the following asymptotic behavior:

$$\bar{\kappa}(\rho;\beta_0) \approx \frac{1}{\sqrt{2\pi}\beta_0}\frac{1}{\rho^2} \sim \rho^{-2}, \quad \rho \gtrsim \beta_0^{-2}. \quad (49)$$

Thus, the exponent α, defined in Eq. (30), changes from the value $\alpha=1/2$ for the idealized balance condition into $\alpha=1$ in the presence of fluctuations of the time deficit parameter from individual to individual and/or as a function of time. The mechanism acting here is similar to the mechanism of "sweeping of an instability" [23], since the presence of a distribution of time deficit parameters around the balance condition $\beta=0$ indeed amounts to sweeping the control parameter β over its bifurcation point defined in Sec. II D. We stress that this "renormalization" of the exponent α from the value 1/2 to 1 is not sensitive to the details of the shape (46) of the distribution of the time deficit parameter β. The single essential feature is that $\psi(\beta;\beta_0)$ goes to a constant for $\beta \to 0$. For any distribution $\psi(\beta;\beta_0)$ having this property of going to a nonzero constant as $\beta\to 0$, the asymptotic tail (49)

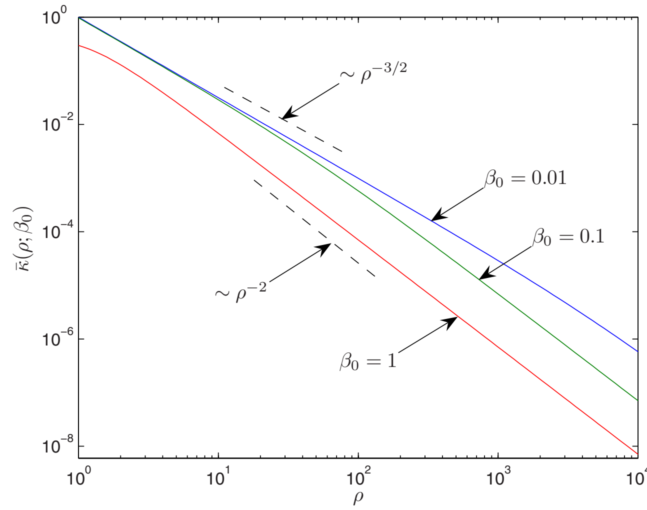

FIG. 5. (Color online) Dependence of the normalized pdf $\bar{\kappa}(\rho;\beta_0)$ given by expression (48) of the waiting times until the completion of the target task as a function of the normalized time ρ for $\beta_0=0.01;0.1;1$. For the smaller values of β, one can observe the intermediate asymptotic law $\sim\rho^{-3/2}$ progressively crossing over to the asymptotic power law tail $\sim\rho^{-2}$.

holds. We will discuss in Sec. III B variations to this conditions and derive the corresponding changes in the exponent α.

For the specific form (46), the above asymptotic result (49) can be made more accurate as follows. For $\beta_0 \gtrsim 1$, $\bar{\kappa}(\rho;\beta)$ represents the unique power law regime (49) with exponent $\alpha=1$. For $\beta_0 \ll 1$, there is an additional intermediate asymptotic power law with exponent $\alpha=1/2$ in the interval [analogous to Eq. (42)]

$$1 \lesssim \rho \lesssim \beta_0^{-2}, \quad (50)$$

which is followed beyond the crossover point $\rho_* \approx \beta_0^{-2}$ by the power law (49) with exponent $\alpha=1$. Figure 5 shows the dependence of $\bar{\kappa}(\rho;\beta_0)$ as a function of ρ given by expression (48) for three different values of β_0 that illustrate the intermediate asymptotic regime with $\alpha=1/2$ and the tail asymptotic regime with $\alpha=1$.

3. Derivation of the survival distribution $Q(t)$ for a Gaussian and a semi-Gaussian distribution of the normalized time deficit parameter β

In the presence of a distribution of time deficit parameters β, the complementary cumulative distribution $Q(t)$ of the waiting time until the completion of the target task can be written as

$$\bar{Q}(\rho;\beta_0) = \frac{1}{\sqrt{2\pi}\beta_0}\int_{-\infty}^{\infty} Q(\rho;\beta)\exp\left(-\frac{\beta^2}{2\beta_0^2}\right)d\beta. \quad (51)$$

Figure 6 plots of the dependence of $\bar{Q}(\rho;\beta_0)$ as a function of ρ for various values of β_0. For all nonzero values of β_0, one can observe that the asymptotic tail exhibits an upward curvature, leading to a departure from the *a priori* expected dependence $\bar{Q}(\rho;\beta_0)\sim\rho^{-1}$. Also the larger the value of β_0,

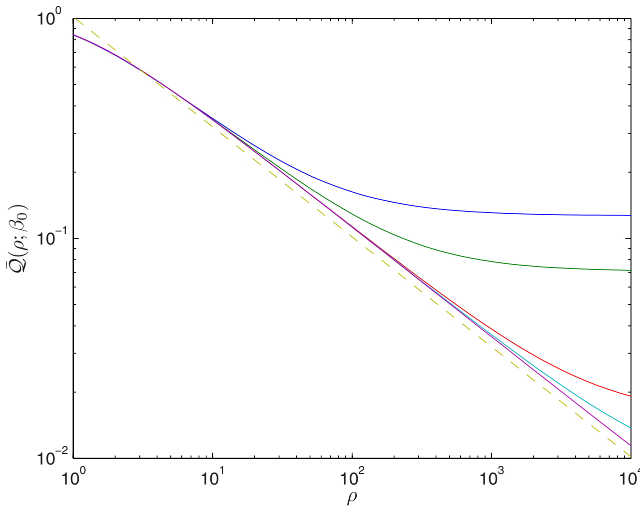

FIG. 6. (Color online) Dependence of the averaged cumulative distribution $\overline{Q}(\rho;\beta_0)$ as a function of the normalized time ρ. Solid lines correspond to $\beta_0=0.1;0.05;0.01;0.005;0.001$ (top to bottom). Dashed straight line is the asymptotic power law (41).

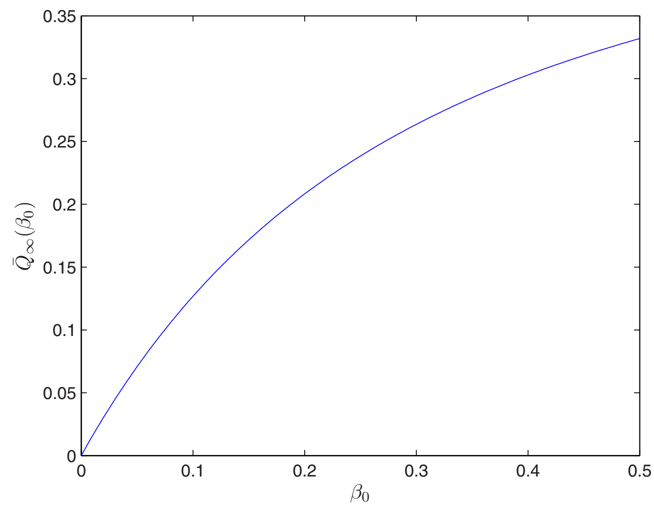

FIG. 7. (Color online) Plot of the limiting probability $\overline{Q}_\infty(\beta_0)$ preventing shaping of the power law of the average complementary distribution $\overline{Q}(\rho;\beta_0)$ for $\rho\to\infty$

the slower the decay of $\overline{Q}(\rho;\beta_0)$, which becomes even slower than $\sim\rho^{-1/2}$.

The origin of the contradiction between the well-defined asymptotic power law (49) for the pdf and the behavior shown in Fig. 6 stems from the existence of the transition occurring at $\beta_0=0$ above which $Q(\rho;\beta)$ acquires a nonzero limit $Q_\infty(\beta)$ given by Eq. (40) at $\rho\to+\infty$, as explained in Sec. II D. Accordingly, the average complementary distribution $\overline{Q}(\rho;\beta_0)$ exhibits the strictly positive limit

$$\lim_{\rho\to\infty}\overline{Q}(\rho;\beta_0)=\bar{Q}_\infty(\beta_0)=\int_0^\infty Q_\infty(\beta)\psi(\beta;\beta_0)d\beta. \quad (52)$$

For the Gaussian distribution (46), this limit is given by

$$\overline{Q}_\infty(\beta_0)=\frac{1}{2}(1-e^{8\beta_0^2}\,\mathrm{erfc}(2\sqrt{2}\beta_0)). \quad (53)$$

Figure 7 shows the dependence of this limit $\overline{Q}_\infty(\beta_0)$ as a function of β_0. In the context of browser updating and software patching, this predicts a regime in which an intermediate asymptotic law $\overline{Q}(\rho;\beta_0)\sim\rho^{-1}$ is followed by a slow crossover to a positive plateau, corresponding to a finite fraction of the population that never upgrades or patches.

In Sec. III A 1, we argued that individuals confronted with a flow rate $\sim 1/\langle\tau\rangle$ of tasks and the desire to solve them characterized by the average solution time $\langle\eta\rangle$ tend to adjust $\langle\tau\rangle$ towards $\langle\eta\rangle$ and/or vice versa. Here, let us consider the possibility that $\langle\tau\rangle$ remains marginally smaller than $\langle\eta\rangle$ so that tasks do not accumulate. Taking into account the heterogeneity of humans and the variability with time of their strategy, this corresponds to changing the Gaussian distribution (46) into the semi-Gaussian distribution

$$\psi_-(\beta;\beta_0)=\begin{cases}0, & \beta>0\\ \sqrt{\dfrac{2}{\pi}}\dfrac{1}{\beta_0}\exp\left(-\dfrac{\beta^2}{2\beta_0^2}\right), & \beta<0.\end{cases} \quad (54)$$

The corresponding average complementary cumulative distribution of the waiting times until the completion of the target task is

$$\overline{Q}_-(\rho;\beta_0)=\sqrt{\frac{2}{\pi}}\frac{1}{\beta_0}\int_{-\infty}^0 Q(\rho;\beta)\exp\left(-\frac{\beta^2}{2\beta_0^2}\right)d\beta. \quad (55)$$

Since the contributions of nonzero values of $\overline{Q}_\infty(\beta_0)$ are removed by this specification of the distribution of β, $\overline{Q}_-(\rho;\beta_0)$ exhibits a well-defined asymptotic power law $\sim\rho^{-1}$. Figure 8 shows the function $\overline{Q}_-(\rho;\beta_0)$ as a function of ρ for different β_0 values and illustrates the crossover from the power law $\sim\rho^{-1/2}$ for $\rho<\beta_0^{-2}$ [condition (50)] to $\sim\rho^{-1}$ at large times ρ.

B. Nonregular distribution of the normalized time deficit parameter β around the origin

The assumption of a Gaussian or semi-Gaussian pdf (46) for the distribution of the normalized time deficit parameter β is representative of the general class of distributions which are regular close to the origin, i.e., converge to a nonzero constant for $\beta\to 0$. As we showed in previous sections, it is the regular behavior around $\beta=0$ which controls the tail of the pdf and survival distribution of waiting times. It is therefore interesting to investigate the consequence of the existence of fewer or more probable deviations from the critical value $\beta=0$. For instance, as mentioned in Sec. III A 1, if individuals adjust their time deficit parameter $\beta\propto\langle\eta\rangle-\langle\tau\rangle$ via state-dependent control actions, we can expect deviations from the assumption that the fluctuations of β are smooth around $\beta=0$. We capture such possibility by considering the following asymptotic behavior of the distribution of β's:

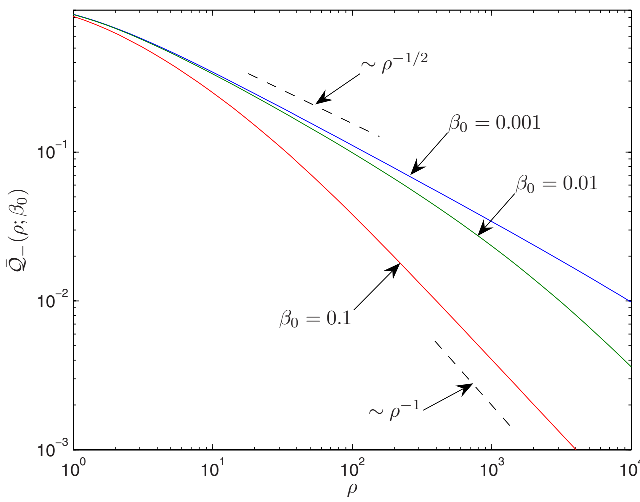

FIG. 8. (Color online) Dependence of the complementary cumulative distribution $\overline{\mathcal{Q}}_-(\rho;\beta_0)$ given by Eq. (55) as a function of the normalized waiting time ρ until the completion of the target task. Top to bottom: β_0=0.001;0.01;0.1. One can observe the crossover from $\sim\rho^{-1/2}$ to $\sim\rho^{-1}$ for β_0=0.01 and the unique power law $\sim\rho^{-1}$ for β_0=0.1

$$\psi_-(\beta;\beta_0|\nu) \approx A(\beta_0,\nu)|\beta|^{\nu}, \quad -1<\nu<+\infty,$$
$$\beta \to -0, \quad (A<\infty). \tag{56}$$

The case ν=0 recovers the regime of Sec. III A. In order to remove the impact of the supercritical domain $\beta>0$ on the survival distribution, we also assume here that all β's are negative. It is a simple matter to remove this condition and recover a regime of nonzero asymptotic behavior for the survival distribution, as discussed in Sec. III A 3.

The corresponding pdf of the waiting time until completion of the target task is given by

$$\bar{\kappa}(\rho;\beta_0|\nu) = \int_{-\infty}^{0} \kappa(\rho;\beta)\psi_-(\beta;\beta_0|\nu)d\beta. \tag{57}$$

In order to determine its asymptotic behavior for large ρ's, it is sufficient to replace this expression by

$$\bar{\kappa}(\rho;\beta_0|\nu) \approx A(\beta_0,\nu)\int_{-\infty}^{0} \kappa(\rho;\beta)|\beta|^{\nu}d\beta, \tag{58}$$

which, using Eq. (38), yields

$$\bar{\kappa}(\rho;\beta_0|\nu) \approx A(\beta_0,\nu)\frac{1}{2}\Gamma\left(\frac{1+\nu}{2}\right)\rho^{-\alpha(\nu)-1},$$
$$\alpha(\nu) = 1+\frac{\nu}{2}, \quad \rho\to\infty. \tag{59}$$

Similarly, the survival distribution of waiting times is given by

$$\overline{\mathcal{Q}}(\rho;\beta_0|\nu) = \int_{-\infty}^{0} \mathcal{Q}(\rho;\beta)\psi_-(\beta;\beta_0|\nu)d\beta \sim \rho^{-\alpha(\nu)}. \tag{60}$$

Since $\nu\in(-1,\infty)$, for $\psi_-(\beta;\beta_0|\nu)$ to be normalized, formula (56) shows that the exponent α can only take values in the interval

$$\alpha \in \left(\frac{1}{2},\infty\right). \tag{61}$$

For instance, Dübendorfer *et al.* [19] reported a value α =2/3 for the decay of the fraction of computers that still keep an outdated Firefox 2 browser. Within the present framework, this corresponds to ν=−2/3, i.e., to a significantly stronger concentration of β close to 0 than would be expected from a semi-Gaussian distribution, for instance.

IV. THEORETICAL FORMULATION OF THE IMPACT OF PROCRASTINATION: NEW POWER LAW REGIMES

In the previous sections, we have assumed that, as soon as all other tasks are solved, the individual addresses without delay the target task with the lowest priority that now comes to the front. In the present section, we explore the consequences of the different possibility that procrastination kicks in so that the target task is postponed and delayed needlessly due to carelessness or laziness or for some other reason.

A. Model and mathematical solution

Consider the flow of new tasks occurring at the times given by Eq. (4) and the process $V(k)$ defined in Eq. (10). Let us denote the times when $V(k)$ touches the value $-\eta_0'$ from above by t_n (see Fig. 2) when the individual is freed from all tasks except the final target task. The set $\{t_n\}$ are the beginnings of the time intervals in which the individual is free to address the target task. Let $N(t)$ be the random number of such free moments in the time interval $(0,t)$ and let us call

$$P(n;t) = \Pr\{N(t)=n\} \tag{62}$$

the probability that the number of spare times in $(0,t)$ is exactly equal to n. We assume that the individual will procrastinate in such a free moment with probability $0\le z<1$. This is the probability of an individual not upgrading their browser or not patching their software in one of their free times. For simplicity, we consider z to be independent of the duration of the free time interval. It would not be difficult to consider alternative specifications dependent on the duration of the free time interval but, for most reasonable versions, the main result of the power law tails obtained below is not modified. Assuming that procrastination is independent in successive free time intervals, the probability that the individual does not complete the target task until time t is given by

$$Q(t,z) = \Pr\{\mathcal{T}>t\} = \sum_{n=0}^{\infty} P(n;t)z^n, \tag{63}$$

where $\mathcal{T}$ is the waiting time until the target task is completed.

In order to calculate $Q(t,z)$ given by Eq. (63), we need the expression of $P(n;t)$. For this, we relate it to the probability

$$F(t;n) = \Pr\{t_n < t\}, \tag{64}$$

which, for a given n, the random variable t_n does not exceed t. The relation between $P(n;t)$ and $F(t;n)$ is

$$P(n;t) = F(t;n) - F(t;n+1) \quad (n \geq 1), \quad P(0;t) = 1 - F(t;1). \tag{65}$$

Expression (65) states that the number n of free intervals occurring in $(0,t)$ is determined by the condition that the nth free time interval starts before t while the $(n+1)$th free time interval starts after t.

Substituting Eq. (65) in relation (63) yields

$$Q(t,z) = 1 + (z-1)\sum_{n=1}^{\infty} F(t;n) z^{n-1}. \tag{66}$$

It is more convenient to work with the pdf of the waiting time until the completion of the target task, defined by $q(t,z) = -\frac{\partial Q(t,z)}{\partial t}$. From Eq. (66), we obtain

$$q(t,z) = (1-z)\sum_{n=1}^{\infty} f(t;n) z^{n-1}, \tag{67}$$

where

$$f(t;n) = \frac{\partial F(t;n)}{\partial t} \tag{68}$$

is the pdf of the random variable t_n, the beginning of the nth free time interval.

As it should, the limit $z=0$ in Eq. (67) recovers the pdf given by Eq. (24) of the waiting time until the completion of the target tasks

$$q(t,0) = f(t;1) \equiv f_1(t). \tag{69}$$

The beginning time t_n of the nth free interval can be written as the sum of n waiting times

$$t_n = \Delta t_1 + \cdots + \Delta t_n. \tag{70}$$

The first waiting time Δt_1 is the duration of the time interval starting at the inception time $t=0$ of the target task until the individual is free to address the target task for the first time. We denote its pdf as $f_1(t)$. The other terms $\Delta t_2, \ldots, \Delta t_n$ quantify the waiting times between successive beginnings of free time intervals. They are independent identically distributed random variables, with a common pdf, denoted $f(t)$. The expressions for $f_1(t)$ and $f(t)$ are made explicit in Sec. IV D. Then, the pdf of the sum (70) is equal to the n-times convolution

$$f(t;n) = f_1(t) \otimes \underbrace{f(t) \otimes \cdots \otimes f(t)}_{n-1 \text{ times}}. \tag{71}$$

The Laplace transform of $f(t;n)$ is thus

$$\hat{f}(s;n) \equiv \int_0^{\infty} f(t;n) e^{-st} dt = \hat{f}_1(s) \hat{f}^{n-1}(s), \tag{72}$$

where $\hat{f}_1(s)$ and $\hat{f}(s)$ are, respectively, the Laplace transforms of $f_1(t)$ and $f(t)$.

Applying the Laplace transform to both sides of equality (67), we obtain

$$\hat{q}(s,z) = (1-z)\hat{f}_1(s)\sum_{n=1}^{\infty} \hat{f}^{n-1}(s) z^{n-1} = \frac{(1-z)\hat{f}_1(s)}{1 - z\hat{f}(z)}. \tag{73}$$

As shown in Sec. IV D, the pdfs $f_1(t)$ and $f(t)$ have in general the following asymptotic power law form:

$$f_1(t) \simeq a_1 t^{-\alpha-1}, \quad f(t) \simeq a t^{-\alpha-1}, \quad t \to \infty. \tag{74}$$

This implies that their Laplace transforms have the following asymptotic form:

$$\hat{f}_1(s) \simeq 1 + \Gamma(-\alpha) a_1 s^{\alpha}, \quad \hat{f}(s) \simeq 1 + \Gamma(-\alpha) a s^{\alpha}, \quad s \to 0. \tag{75}$$

Substituting these last relations into Eq. (73) yields the asymptotic form of $\hat{q}(s,z)$,

$$\hat{q}(s,z) \simeq \frac{1}{1 - \chi s^{\alpha}}, \quad s \to 0, \tag{76}$$

where

$$\chi = -a\Gamma(-\alpha)\frac{z}{1-z} \quad (\chi > 0). \tag{77}$$

B. Expression of the probability z of not completing the target task in one of the free times

Let us denote the duration of the kth free interval, from the beginning time t_k to the arrival of the first new task, by ζ_k. Note that one cannot interpret ζ_k as the duration of the time interval during which the Wiener process $V(t)$ remains below the level $-\eta_0'$ because the formulation in terms of a Wiener process has a sense only for $V(t) > -\eta_0'$. Actually, ζ_k has the simple interpretation of being the waiting time counted from any arbitrary time until the occurrence of a new task.

Consider the simple instance in which the tasks arrive according to a Poisson flow with rate λ, such that the pdf of ζ_k reads

$$\pi(\zeta) = \lambda e^{-\lambda\zeta}. \tag{78}$$

One can interpret $1/\lambda$ as the mean waiting time between task arrivals. For the sake of clarity, let us also assume that the probability of not performing the target task during a free time is a decreasing exponential function of the duration ζ_k of that free interval

$$P(\zeta) = e^{-\lambda_1\zeta}. \tag{79}$$

One can interpret $1/\lambda_1$ as the average "procrastination time." Expression (79) assumes that the individual decides to perform the target task during one of her free times according to a constant probability per unit time, i.e., according to another Poisson process with rate λ_1. Averaging this probability over the statistics of ζ yields the probability z that the target task will not be performed during a given free time interval

$$z = \int_0^\infty P(\zeta)\pi(\zeta)d\zeta = \int_0^\infty e^{-\lambda_1\zeta}\lambda e^{-\lambda\zeta}dz = \frac{\lambda}{\lambda+\lambda_1}. \quad (80)$$

If $\lambda_1 \ll \lambda$, i.e., if the arrival rate of new tasks is significantly larger than the rate with which the individual fights her procrastination, then the probability z of not performing the target task in a given free interval is close to unity. As we see below, this regime $z \to 1$ is responsible for a much slower decay of the pdf and the survival distribution of the waiting times until the completion of the target task.

C. Derivation of the distribution of waiting times until the completion of the target tasks in the presence of pronounced procrastination ($z \to 1$)

It is well known (see, for instance, [24,25]) that the inverse Laplace transform of Eq. (76) is equal to

$$q(t,z) = \frac{1}{\chi^{1/\alpha}}\kappa\left(\frac{t}{\chi^{1/\alpha}},\alpha\right), \quad (81)$$

where $\kappa(y,\alpha)$ can be expressed as the weighted sum of exponential distributions

$$\kappa(y,\alpha) = \int_0^\infty \frac{1}{\mu}\exp\left(-\frac{y}{\mu}\right)\zeta(\mu,\alpha)d\mu, \quad (82)$$

with weights

$$\zeta(\mu,\alpha) = \frac{1}{\pi\mu}\frac{\sin(\pi\alpha)}{\mu^{\alpha}+\mu^{-\alpha}+2\cos(\pi\alpha)}. \quad (83)$$

The corresponding complementary distribution is

$$\mathcal{K}(y,\alpha) \equiv \int_{-\infty}^{y} \kappa(x,\alpha)dx = \int_0^\infty \exp\left(-\frac{y}{\mu}\right)\zeta(\mu,\alpha)d\mu. \quad (84)$$

Expression (82) predicts the existence of the two regimes

$$\kappa(y,\alpha) \sim \frac{1}{y^{1-\alpha}}, \quad \text{for} \quad y \ll 1, \quad \text{and}$$

$$\kappa(y,\alpha) \sim \frac{1}{y^{1+\alpha}}, \quad \text{for} \quad y \gg 1. \quad (85)$$

This translates into

$$q(t,z) \sim \chi^{-1}\frac{1}{t^{1-\alpha}}, \quad \text{for} \quad t \ll \chi^{1/\alpha} \quad \text{and}$$

$$q(t,z) \sim \chi\frac{1}{t^{1+\alpha}}, \quad \text{for} \quad t \gg \chi^{1/\alpha}. \quad (86)$$

Figure 9 plots the pdf $\kappa(y,\alpha)$ given by Eq. (82) for α=0.7 and confirms the existence of an intermediate asymptotic power law regime $q(t,z) \sim 1/t^{1-\alpha}$ for $t \ll \chi^{1/\alpha}$, which decays much slower than in the absence of procrastination [z=0 leading to a small χ and to $q(t,z) \sim 1/t^{1+\alpha}$]. The two regimes can be observed in the data by taking the derivative as a function of time of the fraction of remaining individuals who have not yet solved the target task.

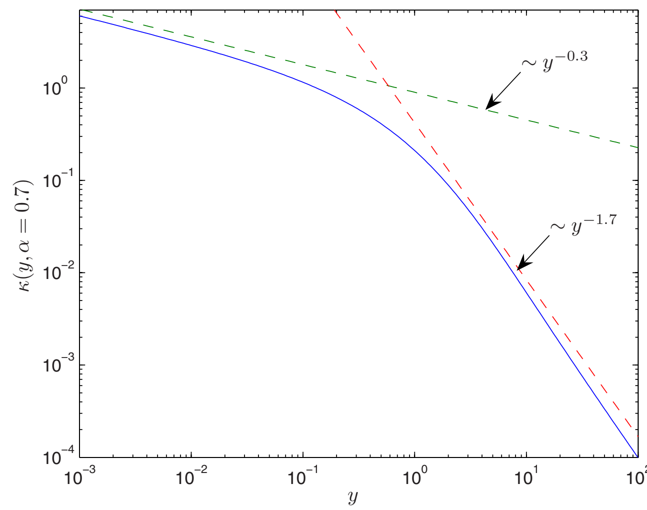

FIG. 9. (Color online) Plot of the pdf $\kappa(y,\alpha)$ given by Eq. (82) as a function of the reduced variable $y=t/\chi^{1/\alpha}$ for α=0.7. The two dashed lines correspond to the intermediate asymptotic regime and to the tail asymptotic law given by Eq. (85).

For $\lambda_1 \ll \lambda$, we can approximate expression (80) by $z=1-\frac{\lambda_1}{\lambda}$, so that χ given by Eq. (77) is approximately equal to $\chi=-a\Gamma(-\alpha)\frac{\lambda}{\lambda_1} \gg 1$, i.e., it is proportional to the ratio of the average "procrastination time" over the mean waiting time between task arrivals. It is this ratio $\frac{\lambda}{\lambda_1}$ that determines the range of the intermediate asymptotic power law $q(t,z) \sim 1/t^{1-\alpha}$, which holds for $t \ll (\lambda/\lambda_1)^{1/\alpha}$.

If we look directly at this fraction [$\mathcal{K}(y,\alpha)$ in normalized units] of nonsolved target task, we do not find two clear power law regimes, but rather a smooth crossover to the asymptotic power law tail $\mathcal{K}(y,\alpha) \sim 1/y^{\alpha}$, as shown in Fig. 10.

D. Derivation of the pdf of the waiting times Δt_k between successive beginnings of free time intervals

Let us now justify the form (74) for the pdf $f_1(t)$ of Δt_1 and for the pdf $f(t)$ of the other independent random variables $\Delta t_2, \ldots, \Delta t_n$ as defined in Eq. (70). As we showed in Sec. II, the pdf of Δt_1 coincides (in the Wiener process approximation) with the pdf for the Wiener process $V(t)$ of first touching the level $-\eta_0'$

$$f_1(t) = \frac{\gamma}{\langle\tau\rangle\sqrt{2\pi}\theta^{3/2}}\exp\left[-\frac{(\delta\theta+\gamma)^2}{2\theta}\right], \quad \theta = \frac{t}{\langle\tau\rangle}. \quad (87)$$

This has the form (74) with α=1/2 when the time deficit parameter δ is close to zero.

The other random variables $\Delta t_2, \ldots, \Delta t_n$ are each the sum of two independent contributions, $\Delta t_k=\zeta_k+\hat{\Delta} t_k$, where (i) ζ_k is the duration of a free time interval, which has the same distribution as that of the waiting time counted from any arbitrary time until the occurrence of a new task and (ii) $\hat{\Delta} t_k$

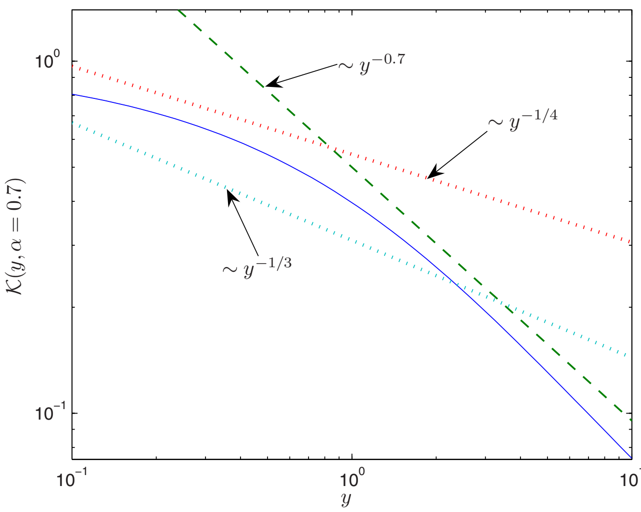

FIG. 10. (Color online) Dependence of the normalized complementary cumulative distribution $\mathcal{K}(y,\alpha)$ given by Eq. (84), for $\alpha=0.7$, as a function of the normalized time $y=t/\chi^{1/\alpha}$. Dashed straight line corresponds to the asymptotic power law $\sim y^{-\alpha}$. The two dotted straight lines are the power laws $\sim y^{-1/3}$ and $\sim y^{-1/4}$ to help the eyes suggest the presence of an intermediate apparent slow power law decay over some limited time range.

is a time similar to Δt_1 for the Wiener process $V(t)$ of first touching some new level. In order to specify further the properties of this second contribution $\hat{\Delta} t_k$, we recall that the pdf $f_1(t)$ of Δt_1 given by Eq. (87) depends on the parameter $\gamma=\frac{\eta'}{\sigma_\eta}$ defined in Eq. (25), where η_0' is the time needed by the individual to solve tasks that has been stored and σ_η is the standard deviations of the times η_k needed to solve the kth task. The pdf $\hat{f}(t)$ of $\hat{\Delta} t_k$ can be written as

$$\hat{f}(t) = \int_0^\infty w(\gamma) f_1(t|\gamma) d\gamma, \qquad (88)$$

where the notation $f_1(t|\gamma)$ makes explicit the dependence on γ in expression (87). The integral in Eq. (88) is performed over the random variable γ weighted by its pdf $w(\gamma)$, which is determined as follows. The end of an interval which was free of any task (except the target task that remains to be addressed) is triggered by the occurrence of a new task that takes priority over the target task and the pdf of the time needed to solve it is $\varphi(\eta)$, with mean and variance equal to $\langle\eta\rangle$ and σ_η^2, as defined in Sec. II B. We thus have

$$w(\gamma) = \sigma_\eta \varphi(\sigma_\eta \gamma). \qquad (89)$$

To illustrate, suppose that $\varphi(\eta)$ is exponential, $\varphi(\eta)=\mu e^{-\mu\eta}$, leading to $\sigma_\eta=1/\mu$ and

$$w(\gamma) = e^{-\gamma}. \qquad (90)$$

Substituting Eqs. (87) and (90) into Eq. (88) yields

$$\hat{f}(t) = \frac{1}{2\langle\tau\rangle}\exp\left(-\frac{\delta^2\theta}{2}\right)\left[\sqrt{\frac{2}{\pi\theta}} - \exp\left(-\frac{(1+\delta)^2\theta}{2}\right) \times(1+\delta)\text{erfc}\left(\frac{(1+\delta)\sqrt{\theta}}{\sqrt{2}}\right)\right]. \qquad (91)$$

It is straightforward to confirm that this pdf $\hat{f}(t)$ has the asymptotic power law (74) with $\alpha=1/2$. Now, $f(t)$ is the convolution of the pdf $\pi(\zeta)$ and of $\hat{f}(t)$ and its tail is determined by that of $\hat{f}(t)$, hence the form (74) with $\alpha=1/2$.

V. CONCLUDING REMARKS

We have developed a simple general framework to model the distribution of waiting times between the triggering factor and the actual realization of a job, for the particular tasks that are both important (sometimes even essential) but are often considered low priority because they require interrupting the normal flow of work or life. Beyond the examples of internet browser updates and software vulnerability patching which initially motivated our interest in this question, we suggest that our theory can apply to less quantifiable but equally important questions such as the delay in implementing important decisions in one's life. While we recognize of course the existence of additional important psychological factors and social influences, our approach provides a simple parsimonious starting point for a general theory of procrastination.

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